Quantized Average Consensus on Gossip Digraphs with Reduced Computation[1]

Kai Cai * and Hideaki Ishii *

Abstract: The authors have recently proposed a class of randomized gossip algorithms which solve the distributed averaging problem on directed graphs, with the constraint that each node has an integer-valued state. The essence of this algorithm is to maintain local records, called "surplus", of individual state updates, thereby achieving quantized average consensus even though the state sum of all nodes is not preserved. In this paper we study a modified version of this algorithm, whose feature is primarily in reducing both computation and communication effort. Concretely, each node needs to update fewer local variables, and can transmit surplus by requiring only one bit. Under this modified algorithm we prove that reaching the average is ensured for arbitrary strongly connected graphs. The condition of arbitrary strong connection is less restrictive than those known in the literature for either real-valued or quantized states; in particular, it does not require the special structure on the network called balanced. Finally, we provide numerical examples to illustrate the convergence result, with emphasis on convergence time analysis.

Key Words: randomized gossip algorithms, quantized averaging, distributed consensus.

1. Introduction

The systems control community has witnessed rapidly growing interest in distributed information consensus over networks of dynamic agents [1]–[5]; also see several textbook treatments [6]–[8] and references therein. The objective of the consensus problem is to design distributed algorithms by which individual nodes can iteratively update their states so as to reach an agreement on some common value. As such a problem finds a broad range of potential applications including motion coordination of multi-robot systems [9], synchronization of coupled oscillators [10], information fusion in sensor networks [11], and load balancing in clusters of processors [12]. The *distributed averaging problem* is of a special form, with the aim of decentrally computing the average of the states of all nodes [13]–[15].

Randomized gossip algorithms have recently been exploited as a stochastic approach to tackle the distributed averaging problem [16]–[20]; refer also to [21] for related problems in search engines. In each iteration of such algorithms, a node communicates with a randomly chosen neighbor, and updates its own value accordingly. Thus the communication and computation burden at individual nodes is relatively low. In addition, gossip iteration provides a natural way of modeling asynchronous behavior of the nodes in a distributed network.

In [16]–[20] the graph is assumed to be undirected, and gossip algorithms are designed such that the state sum of all nodes remains invariant at each iteration, a key property that guarantees converging to the average. By contrast, the authors have recently proposed in [22] a class of *quantized gossip algorithms* (**QA**) for averaging on directed graphs (or digraphs), with an additional constraint that each node has an integer-valued state; the latter follows the work of [18], which potentially models quantization of both stored and transmitted states. The essence of this algorithm is to maintain local records of individual state updates, thereby achieving average consensus even though the state sum is not preserved. We do this by associating each node with an additional variable, called "surplus". It is then proved that reaching the average is assured for arbitrary strongly connected digraphs. The condition of arbitrary strong connection is less restrictive than those in the literature for either real-valued or quantized states (e.g., [13],[15]) in the sense that it does not require maintaining balanced network structure.

In the present paper we study a modified version of this algorithm, referred to as **MQA**, whose feature is primarily in reducing both computation and communication efforts. For computation, each node does not need to update two local variables, which are used in **QA** and termed "local minimum and maximum" [22], but the state set is still guaranteed to be finite. For communication, the surplus can be transmitted by requiring merely one bit, whereas two bits are required in **QA** [22]. The main contribution of this paper is the justification that with reduced processing load on individual nodes, average consensus is again achieved for arbitrary strongly connected digraphs. Moreover, we provide a set of numerical examples to demonstrate the convergence result, with emphasis on convergence time analysis. This paper is based on its conference precursor [23], and we will present further numerical investigations.

The rest of the paper is organized as follows. In Section 2 we formulate the quantized averaging problem on digraphs. In Section 3 we first present the modified gossip algorithm, and then prove the convergence result. Illustrative numerical examples are provided in Section 4; and finally, our conclusions are stated in Section 5.

* The authors are with the Department of Computational Intelligence and Systems Science, Tokyo Institute of Technology, Yokohama 226-8502, Japan.
E-mail: caikai@sc.dis.titech.ac.jp, ishii@dis.titech.ac.jp
(Received August 5, 2010)
[1] This work was supported in part by the Ministry of Education, Culture, Sports, Science and Technology, Japan, under the Grant-in-Aid for Young Scientists(B) Grant No. 21760323.

2. Quantized Averaging Problem

Consider a digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{1, ..., n\}$ is the node set, and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ the edge set. Each directed edge (j, i) in $\mathcal{E}$, pointing from j to i, denotes that i is a neighbor of j and thus j communicates to i. Notice that the information flow over the edge (j, i) is only from j to i.

At time instant $k \in \mathbb{Z}_+$ (nonnegative integers) we assume that each node has an integer-valued state $x_i(k) \in \mathbb{Z}$, $i \in \mathcal{V}$; the aggregate state is denoted by $x(k) = [x_1(k) \cdots x_n(k)]^T \in \mathbb{Z}^n$. We will design randomized gossip algorithms, with which every node updates its state such that all $x_i(k)$ eventually converge to the integer approximation of the initial average. Formally, let $S := x(0)^T \mathbf{1}$, where $\mathbf{1} = [1 \cdots 1]^T$ is the vector of ones. Hence the average of the initial states is S/n, a number that need not be an integer in general. We can, however, always write $S = nL + R$, where L and R are both integers with $0 \leq R < n$. Thus, either L or $L + 1$ (the latter if $R > 0$) may be viewed as an integer approximation of the average S/n. In this paper we consider the convergence to L only.

As in [22], the state sum $x^T \mathbf{1}$ cannot be preserved in general when designing gossip algorithms on digraphs. This is because at each iteration, communication is only unidirectional; thus only one node can receive information and update its state. To tackle this problem, following [22] we associate each node with an additional variable, called surplus, to record the changes in individual states; then the nodes communicate the surplus to their neighbors such that this information can be utilized for state updates. Formally, for every $i \in \mathcal{V}$ let the surplus of node i at time k be $s_i(k) \in \mathbb{Z}$; thus the aggregate surplus is $s(k) = [s_1(k) \cdots s_n(k)]^T \in \mathbb{Z}^n$, the initial value of which is set to be $s(0) = [0 \cdots 0]^T$. As described, the surplus is introduced so as to make the quantity $(x + s)^T \mathbf{1}$ invariant during each iteration, i.e., for each $k \geq 0$,

$$(x(k) + s(k))^T \mathbf{1} = (x(0) + s(0))^T \mathbf{1} = nL + R. \tag{1}$$

Consequently, $s^T \mathbf{1} = R\ (\geq 0)$ if $x = L\mathbf{1}$. Thus we define the set $\mathscr{A}_L$ of average consensus states, which is a subset of $\mathbb{Z}^n \times \mathbb{Z}^n$, by

$$\mathscr{A}_L := \{(x, s) : x_i = L \ \&\ s_i \geq 0,\ i \in \mathcal{V}\}. \tag{2}$$

Definition 1 A network of nodes achieves *quantized average consensus almost surely* if for every initial condition $(x(0), 0)$, there exist a finite time K and $(x^*, s^*) \in \mathscr{A}_L$ such that $(x(k), s(k)) = (x^*, s^*)$ for all $k \geq K$ with probability one.

Finally, we are ready to state the *Quantized Averaging Problem*: Design randomized gossip algorithms and find suitable graphical connectivity such that the nodes achieve quantized average consensus almost surely.

3. Main Result

In this section, we first propose a modified version **MQA** of the quantized gossip algorithm in [22], with the purpose of reducing computation and communication effort. Updating the states of the nodes by **MQA**, we then prove that quantized average consensus can be achieved almost surely for arbitrary strongly connected digraphs.

3.1 *MQA* Algorithm

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ denote a network of nodes. At each time instant $k \in \mathbb{Z}_+$, exactly one edge $(j, i) \in \mathcal{E}$ is activated independently from all earlier instants and with a (time-invariant) strictly positive probability $p_{ji} \in (0, 1)$ such that $\sum_{(j,i)\in\mathcal{E}} p_{ji} = 1$.

Along this activated edge (j, i), node j sends to i its state information $x_j(k)$, as well as its surplus $s_j(k)$. While it does not perform any update on its state, node j does always set its surplus to be 0 after transmission, meaning that its surplus, if any, is entirely passed to its neighbor i; that is,

$$x_j(k + 1) = x_j(k), \quad s_j(k + 1) = 0.$$

On the other hand, node i receives the information from j, namely $x_j(k)$ and $s_j(k)$, and performs the following updates:

(R1) If $x_i(k) = x_j(k)$, then there are two cases:
(i) If $s_i(k) + s_j(k) \geq n$, then

$$x_i(k + 1) = x_i(k) + 1, \quad s_i(k + 1) = s_i(k) + s_j(k) - 1.$$

(ii) Otherwise (i.e., $s_i(k) + s_j(k) < n$),

$$x_i(k + 1) = x_i(k), \quad s_i(k + 1) = s_i(k) + s_j(k).$$

(R2) If $x_i(k) < x_j(k)$, there are also two cases:
(i) If $s_i(k) + s_j(k) > 0$, then

$$x_i(k + 1) = x_i(k) + \Delta(k),$$
$$s_i(k + 1) = s_i(k) + s_j(k) - \Delta(k),$$

where $\Delta(k) \in [1, \min\{s_i(k) + s_j(k), x_j(k)\}]$.
(ii) Otherwise (i.e., $s_i(k) + s_j(k) = 0$),

$$x_i(k + 1) = x_i(k), \quad s_i(k + 1) = s_i(k) + s_j(k).$$

(R3) If $x_i(k) > x_j(k)$, then

$$x_i(k + 1) \in [x_j(k), x_i(k)),$$
$$s_i(k + 1) = s_i(k) + s_j(k) - (x_i(k + 1) - x_i(k)).$$

First, some immediate observations are in sequel. (i) Surplus variables are updated such that the quantity $(x + s)^T \mathbf{1}$ stays invariant at each iteration, and hence (1) is satisfied. (ii) Surpluses are decreased in **(R1)** and **(R2)**, and increased in **(R3)**; it is thus easy to see that all surplus variables are nonnegative [1]. As a result, the definition of average consensus set in (2) can be rewritten as $\mathscr{A}_L := \{(x, s) : x_i = L,\ i \in \mathcal{V}\}$. (iii) Updated by **MQA** the states indeed cannot reach consensus at value $L + 1$; since otherwise $x = (L + 1)\mathbf{1}$, there holds $s^T \mathbf{1} = R - n < 0$, which contradicts that all surpluses are nonnegative.

Second, we emphasize the distinctions between **MQA** and **QA**, which may imply tradeoffs when employing one or the other in practice. (i) In **QA** each node is assigned two additional variables to record, at each iteration, respectively the minimal and maximal states among itself and its neighbors. These variables are called local minimum and maximum, which can prevent the states from exceeding the interval of all initial states, thereby ensuring finite state set [22]. By contrast, we will show for **MQA** that the state set is guaranteed to be finite without the local extrema; consequently, computation effort for a total of $2n$ variables is saved. (ii) For **QA** it is shown that converging to the average is not affected even if the surpluses are transmitted

[1] In **QA**, by contrast, surplus variables can be either positive or negative.

one unit at a time [24]. There this means that the transmitted surpluses may take values only in $\{-1, 0, 1\}$, thus requiring two bits for each transmission. An analogous result for **MQA** will be seen to hold as well. However, since there is no negative surplus, each transmission reduces to one bit for only $\{0, 1\}$. (iii) A drawback of **MQA** lies in that the "threshold" value which determines when to update the state by surplus in **(R1)** equals n, the total number of nodes in the network. By [24, Lemma 7] this has to hold in order to guarantee that $\mathscr{A}_L$ is the unique equilibria set for all initial conditions. For **QA**, however, the threshold can take values in the range $\left[\lfloor n/2 \rfloor + 1, n\right]$ [22].

Third, we provide two instances of **MQA**; that is, we specify the updating rules in **(R2)** and **(R3)**.

Algorithm 1 Node i diminishes the state difference by approximately half of the state sum (in **(R2)** provided that there are available surpluses):

(R2) $\Delta(k) = \min\{\left\lceil \frac{x_i(k)+x_j(k)}{2} \right\rceil - x_i(k), s_i(k) + s_j(k)\}$.

(R3) $x_i(k+1) = \left\lfloor \frac{x_i(k)+x_j(k)}{2} \right\rfloor$.

The functions $\lceil\cdot\rceil, \lfloor\cdot\rfloor : \mathbb{R} \to \mathbb{Z}$ are the *ceiling* and *floor* operators, which round a real number to the smallest following or the largest previous integer, respectively.

Algorithm 2 Node i updates by one unit at a time:

(R2) $\Delta(k) = 1$.

(R3) $x_i(k+1) = x_i(k) - 1$.

The verification that these two examples indeed satisfy **MQA** is straightforward.

3.2 Convergence Result

Our main result is the following theorem, a solution for the Quantized Averaging Problem formulated in Section 2.

Theorem 1 Using **MQA** the nodes achieve quantized average consensus almost surely if and only if the digraph $\mathcal{G}$ is strongly connected (i.e., every node is reachable from every other node).

Thus, the necessary and sufficient graphical condition ensuring average consensus for **MQA** is the same as that for **QA** [22]. As mentioned above, however, each node now computes two local variables less than in **QA**, and each surplus transmission requires one bit less. Hence, the contribution of this result is that with reduced processing load on individual nodes, quantized average consensus is achieved without imposing stronger conditions on graph connectivity.

It is also worth noting that without augmenting surplus variables, to reach average consensus on digraphs we need both strongly connected and *balanced* topologies (or, equivalently, *doubly stochastic* system matrices) [13],[15]. A balanced digraph is one where every node has the same number of incoming and outgoing (uniformly weighted) edges. However, this condition can be difficult to be maintained when the communication is asynchronous. By contrast, our condition on digraphs does not require the balanced property; and in fact, only one directed edge is activated at a time.

3.3 Convergence Proof

We now provide the proof of Theorem 1. Two preliminary results will first be established, in which we assume that $\mathcal{G}$ is strongly connected. For their proofs we refer to [24].

Given an arbitrary state $x(k)$, define the minimum and maximum by

$$m(k) = \min_{i \in \mathcal{V}} x_i(k), \ M(k) = \max_{i \in \mathcal{V}} x_i(k). \tag{3}$$

In the case where all nodes have the same state (i.e., $m(k) = M(k)$), our first result ensures that there exists a strictly positive probability such that all surpluses in the system can be 'collected' by a single node in finite time.

Lemma 1 Suppose that at time k (≥ 0), the pair $(x(k), s(k))$ is such that $m(k) = M(k)$. Then for an arbitrary node $i \in \mathcal{V}$, there exists a finite time $K_s > k$ such that

$$\Pr\big[x(K_s) = x(k),\ s_i(K_s) = s_1(k) + \cdots + s_n(k),\\ (\forall j \neq i)\ s_j(K_s) = 0 \mid (x(k), s(k))\big] > 0.$$

Next, recall from (1) that $(x(0) + s(0))^T \mathbf{1} = nL + R$, where $R \in [0, n-1]$. As the quantity $(x+s)^T\mathbf{1}$ is invariant, if all states are identical to $L-\alpha$ for some $\alpha \geq 1$, then the total surplus in the system is $s^T\mathbf{1} = R + \alpha n$. Now suppose that one node i increases its state to $L-\alpha+1$ and has all the surpluses $R+\alpha n-1$. In order to approach the set $\mathscr{A}_L$ defined in (2), it is desired that other nodes 'follow' i to the state $L - \alpha + 1$, thereby decreasing the total surplus to $R + (\alpha - 1)n$. Our second result asserts that this can be done in finite time and with a strictly positive probability.

Lemma 2 Suppose that at time k (≥ 0), the pair $(x(k), s(k))$ is such that for one node $i \in \mathcal{V}$

$$x_i(k) = L - \alpha + 1, \quad s_i(k) = R + \alpha n - 1,$$

and for other nodes $j \neq i$

$$x_j(k) = L - \alpha, \quad s_j(k) = 0.$$

Then there exists a finite time $K_u > k$ such that

$$\Pr\big[m(K_u) = M(K_u) = L - \alpha + 1,\\ s_i(K_u) = R + (\alpha - 1)n,\\ (\forall j \neq i)\ s_j(K_u) = 0 \mid (x(k), s(k))\big] > 0.$$

Proof of Theorem 1: (Necessity) This part is the same as that for **QA**; hence we refer to [22],[24].

(Sufficiency) Based on [18, Theorem 2], it suffices to establish the following three conditions:

(C1) The evolution of $(x(k), s(k))$, $k \geq 0$, is a Markov chain with a finite state space;

(C2) the set $\mathscr{A}_L$ defined in (2) is an invariant set;

(C3) for every $k \geq 0$ there is a finite time $K_a \geq k$ such that $\Pr[(x(K_a), s(K_a)) \in \mathscr{A}_L \mid (x(k), s(k))] > 0$.

For **(C1)**: That the evolution of $(x(k), s(k))$ is Markovian follows directly from the gossip assumption where at time k one edge is activated at random and independently from all earlier instants. Next, we prove that the state space is finite. Since no negative surplus can be generated by **MQA**, the minimum $m(k)$, $k \geq 0$, is non-decreasing. Thus, in turn, for $M(k)$ there is an upper bound $S - (n-1)m(0)$, where $S = x(0)^T \mathbf{1}$. This upper bound may be achieved when there is one node having this value, all other $n-1$ nodes having $m(0)$, and all surpluses equal 0. It then follows that an upper bound for the size of the state set is $(S - nm(0) + 1)^n$. In addition, for surpluses we derive that for every $i \in \mathcal{V}$ and $k \geq 0$, $s_i(k) \in [0, S - nm(0)]$. Thus, an upper bound for the size of the surplus set is also $(S - nm(0) + 1)^n$. Therefore, the state space of $(x(k), s(k))$ is finite.

For **(C2)**: Let $(x(k), s(k)) \in \mathscr{A}_L$, i.e.,

$$(\forall i \in \mathcal{V})\ x_i(k) = L,\ s_i(k) \geq 0,\ \sum_{i=1}^{n} s_i(k) = R.$$

Then for an arbitrary edge $(h, j) \in \mathcal{E}$ activated,

$$\begin{aligned} x_h(k) &= x_j(k), \\ s_h(k) + s_j(k) &\leq \sum_{i=1}^{n} s_i(k) = R < n. \end{aligned}$$

Thus only **(R1)(ii)** of **MQA** applies, and the subsequent states and surpluses satisfy $(x(k'), s(k')) \in \mathscr{A}_L$ for all $k' > k$, i.e., the set $\mathscr{A}_L$ is (positive) invariant.

For **(C3)**: Let $(x(k), s(k))$, $k \geq 0$, be arbitrary. We consider respectively the two cases $m(k) = M(k)$ and $m(k) \neq M(k)$.

1) $m(k) = M(k)$. Notice that necessarily $m(k) = M(k) \in [m(0), L]$, as otherwise the total surplus sum would be negative. If $m(k) = M(k) = L$, then $(x(k), s(k)) \in \mathscr{A}_L$. Letting $K_a = k$ we obtain the conclusion. It is left to consider the case $m(k) = M(k) = L - \alpha$ for some $\alpha \in [1, L - m(0)]$. For this, fix an arbitrary node i and apply Lemma 1; we derive that there exists a finite time $K_0 > k$ such that

$$\begin{aligned} &\Pr[x(K_0) = x(k),\ s_i(K_0) = s_1(k) + \cdots + s_n(k), \\ &\quad (\forall j \neq i)\ s_j(K_0) = 0 \mid (x(k), s(k))] > 0. \end{aligned}$$

Hence with a strictly positive probability, $s_i(K_0) = R + \alpha n$. As the digraph $\mathcal{G}$ is strongly connected, there must exist another node $j \neq i$ with an edge $(j, i) \in \mathcal{E}$. Along this edge the following conditions hold:

$$\begin{aligned} &x_i(K_0) = x_j(K_0) = L - \alpha, \\ &s_i(K_0) + s_j(K_0) = R + \alpha n \geq n. \end{aligned}$$

When this edge is activated, **(R1)(i)** of **MQA** applies:

$$\begin{aligned} &x_i(K_0 + 1) = x_i(K_0) + 1 = L - \alpha + 1, \\ &s_i(K_0 + 1) = s_i(K_0) + s_j(K_0) - 1 = R + \alpha n - 1. \end{aligned}$$

Now the conditions of Lemma 2 are met; we hence obtain that there exists a finite time $K_1 > K_0 + 1$ such that

$$\begin{aligned} &\Pr[m(K_1) = M(K_1) = L - \alpha + 1, \\ &\quad s_i(K_1) = R + (\alpha - 1)n, \\ &\quad (\forall j \neq i)\ s_j(K_1) = 0 \mid (x(k), s(k))] > 0. \end{aligned}$$

Repeating the above process, we derive a sequence of times $K_1 < K_2 < \cdots < K_\alpha$, and at the last time K_α,

$$\begin{aligned} &\Pr[m(K_\alpha) = M(K_\alpha) = L,\ s_i(K_\alpha) = R, \\ &\quad (\forall j \neq i)\ s_j(K_1) = 0 \mid (x(k), s(k))] > 0. \end{aligned}$$

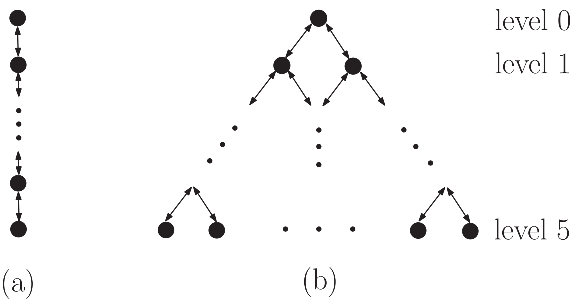

Fig. 1 (a) Line digraph; (b) perfect binary tree digraph.

Set $K_a = K_\alpha$ and **(C3)** holds.

2) $m(k) \neq M(k)$. Write $x(k) = [x_1(k) \cdots x_n(k)]^T$, and choose a node $j \in \mathcal{V}$ such that $x_j(k) \in [m(0), L]$. We can show [24] that under **MQA**, there exists a finite time $\bar{k} > k$ such that

$$\Pr[(x(\bar{k}), s(\bar{k}) = (x_j(k)\mathbf{1}, s(\bar{k})) \mid (x(k), s(k))] > 0.$$

Hence with a strictly positive probability, $m(\bar{k}) = M(\bar{k}) = x_j(k)$. This situation is that in case 1), for which **(C3)** is established.

■

Remark 1 We point out that the necessity and sufficiency proofs hold even if the surpluses are transmitted one unit at a time. In that case, when there is more than one-unit surplus to be transmitted from a node j to another node i (i.e., $s_j > 1$), we may consecutively select the edge (j, i) for communication until all surpluses are transmitted. Such a selection, by our gossip setup, is with a strictly positive probability. As a result, each transmission of surplus requires merely one bit.

4. Numerical Examples

We provide a set of numerical examples to demonstrate the convergence result of **MQA**, the impact of graph connectivity on convergence time, and also the influence of network size on convergence time. In all examples, the states of the nodes are randomly initialized from a uniform distribution on the interval $[-5, 5]$. In addition, we suppose that every edge has the same probability of being activated (i.e., p_{ji} are equal for all $(j, i) \in \mathcal{E}$), and that the concrete rules in Algorithm 1 are used for simulation.

4.1 Line and Tree Digraphs

First, we display the convergence sample paths of line and tree digraphs, and compare the convergence time in these two different topologies.

In a line digraph, the nodes are arranged into a line configuration, as displayed in Fig. 1 (a) [2]. The two end nodes have only one neighbor, while the other nodes in between have two neighbors. We show the convergence sample path on the line of 63 nodes in Fig. 2; for this example the initial state sum is $\sum_{i=1}^{63} x_i(0) = 16$. We see that the states converge to the quantized average $L = 0$, and the corresponding total surplus settles at 16. Note that the convergence time of this sample path is 4.61×10^4; for 100 runs of **MQA** on the same line digraph, we obtain the average convergence time 4.84×10^4.

[2] In this and subsequent figures, for compactness we draw "↔" to represent *two* unidirectional edges: "←" and "→". Note that in **MQA**, only one directed edge is activated at a time.

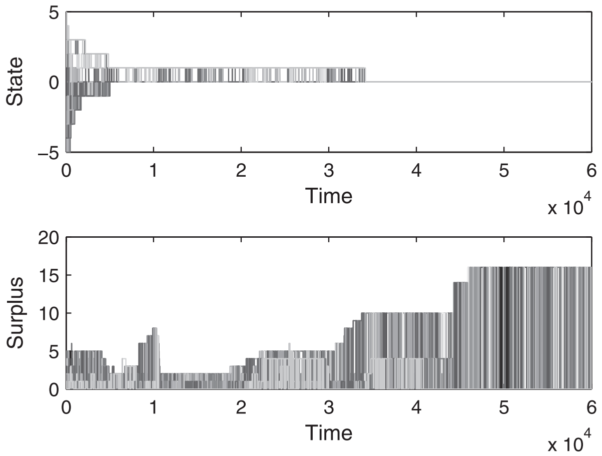

Fig. 2 Convergence of 63 nodes on line digraph.

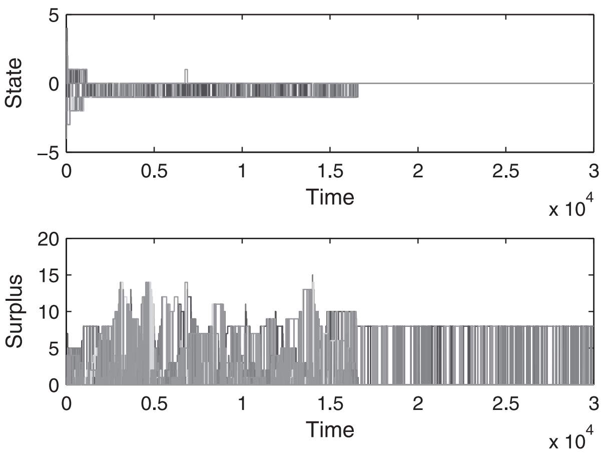

Fig. 3 Convergence of 63 nodes on perfect binary tree digraph.

We now structure the nodes into a *perfect binary tree* (e.g., [25]), in which every level is completely filled. That is, every level d $(= 0, 1, \ldots)$ has 2^d nodes. Since there are 63 nodes, they comprise a perfect binary tree of five levels (see Fig. 1 (b)). Figure 3 shows the convergence sample path on this tree for the initial state sum $\sum_{i=1}^{63} x_i(0) = 8$. All states converge to the average $L = 0$, with the steady state surplus being 8. This convergence takes 1.66×10^4 time steps; also the average of 100 runs of **MQA** on the same tree digraph is 1.95×10^4. Compared to line topology, the convergence time is shorter in tree topology; this is due mainly to the structure of perfect binary trees, which considerably shortens the distance between the farthest-apart nodes. In our examples of 63 nodes, this distance equals 62 in the line digraph, while only 10 in the tree digraph. Hence, the speed of information fusion in tree networks may be accelerated.

4.2 Convergence Time versus Degree of Connectivity

Next, we investigate the impact of graph connectivity on convergence time.

We introduce the types of digraphs that are used for this investigation. A *regular* digraph of degree d $(\in \mathbb{Z}_+)$, denoted by $\mathcal{G}_d$, is a digraph where every node has the same number d of both incoming and outgoing edges (e.g., [25]). We consider a type of regular digraphs $\mathcal{G}_d$ in which each node, placed on a ring, is connected to its nearest d neighbors; see an instance in Fig. 4 (a). In addition, we consider randomizing the choices of neighbors: Instead of being fixed to the nearest, the d neighbors of each node are chosen uniformly at random from the rest (see Fig. 4 (b)). Note that the resulting digraphs will not be regular in general, as different nodes can have different numbers of incoming edges.

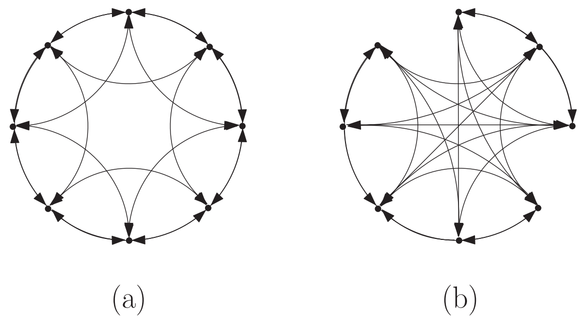

Fig. 4 (a) Example of nearest-neighbor regular digraph: $\mathcal{G}_4$ of 8 nodes; (b) randomized version of (a): for each node in turn, its 4 neighbors are randomly selected.

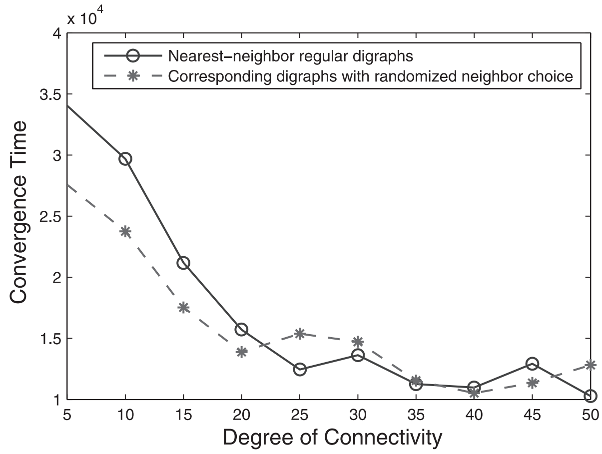

Fig. 5 Convergence time versus degree of connectivity.

Intuitively, one would expect shorter convergence time for higher degree of graph connectivity. To verify this we first consider a series of nearest-neighbor regular digraphs of 51 nodes with increasing degrees: $\mathcal{G}_{5d}, d = 1, 2, \cdots, 10$. We plot in Fig. 5 (solid curve) the average convergence time of 100 runs of **MQA** for each degree. It is observed that when the degree increases in the beginning, the convergence time drops noticeably; thereafter, however, it stays roughly at the same level. We conduct the same experiment on the corresponding digraphs with randomized neighbor choice; the dashed curve in Fig. 5 is obtained, which exhibits similar trends to the solid one.

This phenomenon observed in both cases might suggest that the degree of connectivity, after reaching a certain value (probably related to the number n of nodes), would no longer play a significant role in reducing convergence time. Also, it is worth pointing out that the dashed curve is faster when the degree of connectivity is relatively low (from 5 to 20), owing possibly to the following reason. For low degree of connectivity, individual nodes in nearest-neighbor topology can exchange information only within a short range; in contrast, randomizing neighbor choice may allow communication between nodes far-apart, thereby potentially accelerating the convergence.

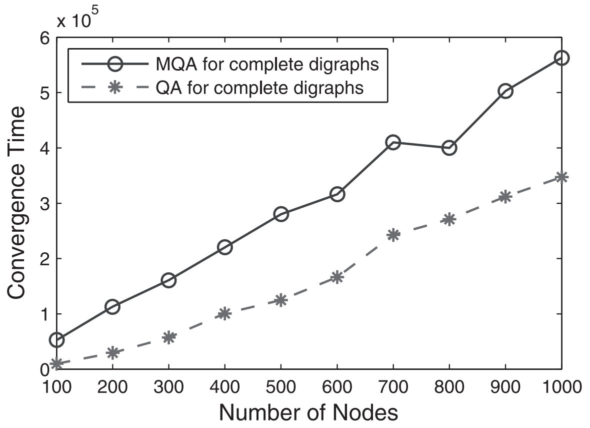

Fig. 6 Convergence time versus number of nodes for both **MQA** and **QA** on complete digraphs.

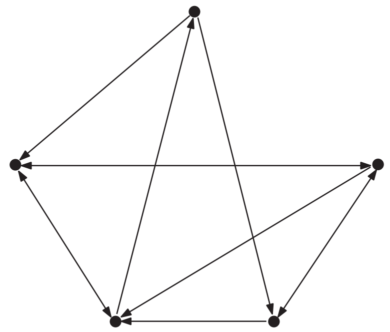

Fig. 7 Example of complete random digraph with 5 nodes and edge existence probability $p = 1/2$.

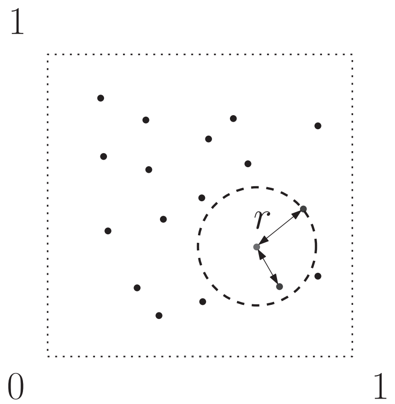

Fig. 8 In a geometric random digraph, each node is connected to those nodes that are within distance r from itself.

4.3 Convergence Time versus Number of Nodes

We turn finally to the study of convergence time with respect to the number of nodes in the network.

First, we deal with the increasing rates of convergence time as the number of nodes increases for both **MQA** and **QA** on complete digraphs (i.e., every node is connected to every other node via a directed edge). The results are respectively the solid and dashed curves in Fig. 6, each plotted value being the average convergence time of 100 runs of the corresponding algorithms. It is seen that both increasing rates are approximately of linear order, the rigorous justification of which will be targeted in our future work. Also the comparison of these two curves indicates the following tradeoff: The benefit of **MQA** in reducing computation and communication effort is at the cost of increasing convergence time.

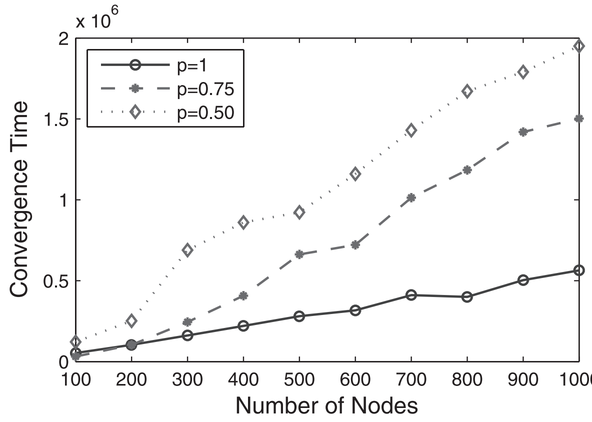

Fig. 9 Convergence time versus number of nodes for **MQA** on complete random digraphs with different choices of probability p.

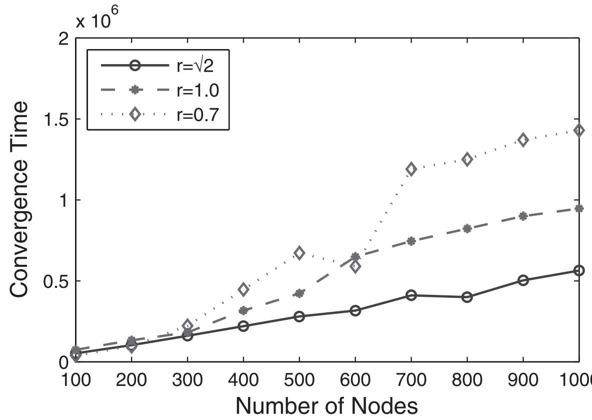

Fig. 10 Convergence time versus number of nodes for **MQA** on geometric random digraphs with different choices of radius r.

Second, we do an analogous investigation for **MQA** on two types of random digraphs. One type is defined as follows (e.g., [26]): The existence of an edge between *every* pair of nodes is determined randomly, independent of other edges, with a (possibly non-uniform) strictly positive probability. Since such type of digraphs is complete in expectation, we refer to it as *complete random digraphs*. Here for simplicity, we assume that every edge exists with the same probability p; see an example in Fig. 7. The other type is the *geometric random digraphs* (e.g., [27]), which have been widely used for modeling *ad hoc* wireless sensor networks. In two dimensions, a geometric random digraph $\mathcal{G}(n, r)$ denotes a network of n nodes whose transmission radius is within r. It is obtained by placing n nodes uniformly at random in a unit square, and connecting every pair of nodes to each other that are within distance r; see the illustration in Fig. 8.

We display the convergence time of **MQA** on complete random digraphs with $p = 0.5, 0.75, 1$ in Fig. 9, and on geometric random digraphs with $r = 0.7, 1, \sqrt{2}$ in Fig. 10. Note that the two cases $p = 1$ and $r = \sqrt{2}$ correspond to complete digraphs. It is observed in both types of random digraphs that the convergence speed becomes slower as the respective parameters p and r decrease, as well as the network expands (i.e., the number n of nodes increases). This indicates that smaller values of the parameters result in lower degrees of graph connectivity, and this effect may be magnified by large-scale networks.

5. Conclusions

We have studied a modified version of a class of randomized gossip algorithms that the authors recently proposed to solve the quantized averaging problem on digraphs. The primary feature of this modification is in reducing both computation and communication efforts on individual nodes. Converging to the quantized average for arbitrary strongly connected digraphs is formally justified, and a set of numerical examples is provided to demonstrate this convergence result, with emphasis on convergence time analysis.

Our future work aims first of all at deriving theoretical bounds on convergence time in terms of the number of nodes in the network. Second, the issue of finding other faster randomized averaging algorithms for digraphs also deserves further investigation. Finally, there is another problem arising if the packets containing surpluses can be lost during transmission. We are interested in extending current algorithms so as to handle this packet loss problem.

References

[1] A. Jadbabaie, J. Lin, and A.S. Morse: Coordination of groups of mobile autonomous agents sing nearest neighbor rules, *IEEE Trans. Autom. Control*, Vol. 48, No. 6, pp. 988–1001, 2003.

[2] Z. Lin, M. Broucke, and B.A. Francis: Local control strategies for groups of mobile autonomous agents, *IEEE Trans. Autom. Control*, Vol. 49, No.4, pp. 622–629, 2004.

[3] L. Moreau: Stability of multi-agent systems with time dependent communication links, *IEEE Trans. Autom. Control*, Vol. 50, No. 2, pp. 169–182, 2005.

[4] R. Olfati-Saber, J.A. Fax, and R.M. Murray: Consensus and cooperation in networked multi-agent systems, *Proc. IEEE*, Vol. 95, No. 1, pp. 215–233, 2007.

[5] L. Xiao and S. Boyd: Fast linear iterations for distributed averaging, *Systems & Control Letters*, Vol. 53, No. 1, pp. 65–78, 2004.

[6] D.P. Bertsekas and J.N. Tsitsiklis: *Parallel and Distributed Computation: Numerical Methods*, Prentice-Hall, 1989.

[7] F. Bullo, J. Cortés, and S. Martínez: *Distributed Control of Robotic Networks*, Princeton University Press, 2009.

[8] W. Ren and R.W. Beard: *Distributed Consensus in Multi-vehicle Cooperative Control: Theory and Applications*, Springer-Verlag, 2008.

[9] J. Cortés, S. Martínez, and F. Bullo: Robust rendezvous for mobile autonomous agents via proximity graphs in d dimensions, *IEEE Trans. Autom. Control*, Vol. 51, No. 8, pp. 1289–1298, 2004.

[10] S.H. Strogatz: From Kuramoto to Crawford: Exploring the onset of synchronization in populations of coupled oscillators, *Physica D*, Vol. 143, pp. 1–20, 2000.

[11] L. Xiao, S. Boyd, and S. Lall: A scheme for robust distributed sensor fusion based on average consensus, *Proc. 4th International Symp. Information Processing in Sensor Networks*, pp. 63–70, 2005.

[12] B. Ghosh and S. Muthukrishnan: Dynamic load balancing by random matchings, *J. Computer and System Sciences*, Vol. 53, pp. 357–370, 1996.

[13] R. Carli, F. Fagnani, A. Speranzon, and S. Zampieri: Communication constraints in the average consensus problem, *Automatica*, Vol. 44, No. 3, pp. 671–684, 2008.

[14] A. Nedic, A. Olshevsky, A. Ozdaglar, and J.N. Tsitsiklis: On distributed averaging algorithms and quantization effects, *IEEE Trans. Autom. Control*, Vol. 54, No. 11, pp. 2506–2517, 2009.

[15] R. Olfati-Saber and R.M. Murray: Consensus problems in networks of agents with switching topology and time-delays, *IEEE Trans. Autom. Control*, Vol. 49, No. 9, pp. 1520–1533, 2004.

[16] S. Boyd, A. Ghosh, B. Prabhakar, and D. Shah: Randomized gossip algorithms, *IEEE Trans. Inf. Theory*, Vol. 52, No. 6, pp. 2508–2530, 2006.

[17] R. Carli, F. Fagnani, P. Frasca, and S. Zampieri: Gossip consensus algorithms via quantized communication, *Automatica*, Vol. 46, No. 1, pp. 70–80, 2010.

[18] A. Kashyap, T. Başar, and R. Srikant: Quantized consensus, *Automatica*, Vol. 43, No. 7, pp. 1192–1203, 2007.

[19] J. Lavaei and R.M. Murray: Quantized consensus via adaptive stochastic gossip algorithm, *Proc. 48th IEEE Conf. Decision and Control*, pp. 5756–5762, 2009.

[20] M. Zhu and S. Martínez: On the convergence time of distributed quantized averaging algorithms, *Proc. 47th IEEE Conf. Decision and Control*, pp. 3971–3976, 2008.

[21] H. Ishii and R. Tempo: Distributed randomized algorithms for the pagerank computation, *IEEE Trans. Autom. Control*, Vol. 55, No. 9, pp. 1987–2002, 2010.

[22] K. Cai and H. Ishii: Gossip consensus and averaging algorithms with quantization, *Proc. 2010 American Control Conf.*, pp. 6306–6311, 2010.

[23] K. Cai and H. Ishii: Further results on randomized quantized averaging: A surplus-based approach, *Proc. 2010 IEEE Multi-Conf. on Systems and Control*, pp. 1558–1563, 2010.

[24] K. Cai and H. Ishii: Quantized consensus and averaging on gossip digraphs, *IEEE Trans. Autom. Control*, to appear, 2011.

[25] C. Godsil and G. Royle: *Algebraic Graph Theory*, Springer, 2001.

[26] Y. Hatano and M. Mesbahi: Agreement over random networks, *IEEE Trans. Autom. Control*, Vol. 50, No. 11, pp. 1867–1872, 2005.

[27] P. Gupta and P.R. Kumar: The capacity of wireless networks, *IEEE Trans. Inf. Theory*, Vol. 46, No. 2, pp. 388–404, 2000.

Kai Cai (Student Member)

He received the B. Eng. degree in electrical engineering from Zhejiang University, China in 2006, and the M.A.Sc. degree in electrical and computer engineering from the University of Toronto, Canada in 2008. He is currently pursuing the Doctoral degree at the Tokyo Institute of Technology, Japan. His research interest is in distributed control of multi-agent systems.

Hideaki Ishii (Member)

He received the M.Eng. degree in applied systems science from Kyoto University in 1998, and the Ph.D. degree in electrical and computer engineering from the University of Toronto in 2002. He was a Postdoctoral Research Associate of the Coordinated Science Laboratory, University of Illinois at Urbana-Champaign from 2001 to 2004, and a Research Associate of the Department of Information Physics and Computing, The University of Tokyo from 2004 to 2007. He is currently an Associate Professor of the Department of Computational Intelligence and Systems Science, Tokyo Institute of Technology. His research interests are in networked control systems, multi-agent systems, hybrid systems, and probabilistic algorithms. He serves as an Associate Editor for Automatica and the IEEE Transactions on Automatic Control.